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SINGULARLY PERTURBED PROBLEMS OF HYPERBOLIC-PARABOLIC TYPE WITH LIPSCHITZIAN NONLINEARITY

We study the behavior of solutions of the Cauchy problem $\varepsilon u''(t)+u'(t)+Au(t)+B(u(t)) = f(t), u(0) = u_0$, $u'(0) = u_1$ in the Hilbert space H as $\varepsilon \to 0$, where A is a linear, self-adjoint, strong positive operator and B is nonlinear Lipschitzian operator.

1. Introduction.

Let V and H be the real Hilbert spaces equipped with the norms $||\cdot||$ and $|\cdot|$, respectively, such that $V \subset H$, where the embedding is defined densely and continuously. By $(,)$ denote the scalar product in H. Let $A : V \to H$ be a linear, self-adjoint operator and

$$(Au, u) \geq \omega ||u||^2, \quad \forall u \in V, \quad \omega > 0. \tag{1}$$

Let $B : H \to H$ be a nonlinear operator which satisfies the Lipschitz condition

$$|B(u) - B(v)| \leq L|u - v|, \quad \forall u, v \in H. \tag{2}$$

In this paper we shall study the behavior of the solutions of the problem

$$\begin{cases} \varepsilon u''(t) + u'(t) + Au(t) + B(u(t)) = f(t), \\ u(0) = u_0,\ u'(0) = u_1 \end{cases} \tag{P_ε}$$

as $\varepsilon \to 0$, where ε is a small positive parameter. Our aim is to show that $u \to v$ as $\varepsilon \to 0$, where v is the solution of the problem

$$\begin{cases} v'(t) + Av(t) + B(v(t)) = f(t), \\ v(0) = u_0, \end{cases} \tag{P_0}$$

The main tool in our approach is the relation between the solutions of the problems (P_ε) and (P_0) in the linear case.

Let us remind some notations which will be used in the sequel.

For $k \in \mathbb{N}, p \in [1, \infty)$ and $(a, b) \subset (-\infty, +\infty)$ we denote by $W^{k,p}(a, b; H)$ the usual Sobolev spaces of vectorial distributions: $W^{k,p}(a, b; H) = \{f \in D'(a, b; H);\ u^{(l)} \in L^p(a, b; H)$, $l = 0, 1, \ldots, k\}$ equipped with the norm

$$||f||_{W^{k,p}(a,b;H)} = (\sum_{l=0}^{k} ||f^{(l)}||^p_{L^p(a,b;H)})^{1/p}.$$

For each $k \in \mathbb{N}, W^{k,\infty}(a, b; H) = \{f \in D'(a, b; H);\ u^{(l)} \in L^\infty(a, b; H), l = 0, 1, \ldots, k\}$ is the Banach space equipped with the norm

$$||f||_{W^{k,\infty}(a,b;H)} = \max_{0 \leq l \leq k} ||f^{(l)}||_{L^\infty(a,b;H)}.$$

For $s \in \mathbb{R}, k \in \mathbb{N}$ and $p \in [1, \infty]$ we denote the following Banach spaces $W^{k,p}_s(a, b; H) = \{f : (a, b) \to H; e^{-st} f^{(l)} \in L^p(a, b; H), l = 0, 1, \ldots, k\}$ equipped with the norm

$$||f||_{W^{k,p}_s(a,b;H)} = \max_{0 \leq l \leq k} ||e^{-st} f^{(l)}(\cdot)||_{L^p(a,b;H)}.$$

2. A priori estimates for solutions of the problem ($\mathbf{P}_\varepsilon$).

In this section we shall prove an *a priori* estimates for the solutions of the problems (P_ε) which are uniform relative to the small values of parameter ε. First of all we shall remind the existence theorem for the solutions of the problems (P_ε) and (P_0).

THEOREM A. [1]. *Let $T > 0$. Suppose that $f \in W^{1,1}(0,T;H)$, $u_0, u_1 \in V$ and the operators A and B satisfy the conditions (1) and (2) respectively. Then there exists a unique function $u \in C(0,T;H) \cap L^\infty(0,T;V)$ satisfying the problem (P_ε) and the conditions: $Au \in L^\infty(0,T;H)$, $u' \in L^\infty(0,T;V)$, $u'' \in L^\infty(0,T;H)$.*

THEOREM B. [1]. *If $f \in W^{1,1}(0,T;H), u_0 \in V$ and A and B satisfy the conditions (1), (2), then there exists a unique strong solution $v \in W^{1,\infty}(0,T;H)$ of the problem (P_0) and the estimates*

$$|v(t)| \leq e^{(L-\omega)t}\Big(|u_0| + \int_0^t e^{-(L-\omega)\tau}(|f(\tau) - B(0)|)d\tau\Big),$$

$$|v'(t)| \leq e^{(L-\omega)t}\Big(|Au_0 + B(u_0) - f(0)| + \int_0^t e^{-(L-\omega)\tau}|f'(\tau)|d\tau\Big),$$

are true for $0 \leq t \leq T$.

We remind that a function $v \in C([0,T];H)$ is said to be a *strong solution* (in the following named *solution*) for Cauchy problem $(P.v)$ if: a)v is absolutely continuous on any compact subinterval of $(0,T)$; b) $v(t) \in D(A)$ a.e. $t \in (0,T)$; c) $v(0) = u_0$ and v satisfies the equation from $(P.v)$ a.e. $t \in (0,T)$.

Before to prove the estimates for solutions of problem (P_ε) we recall the following well-known lemma.

LEMMA A. [2]. *Let $\psi \in L^1(a,b)(-\infty < a < b < \infty)$ with $\psi \geq 0$ a. e. on (a,b) and let c be a fixed real constant. If $h \in C([a,b])$ verify*

$$\frac{1}{2}h^2(t) \leq \frac{1}{2}c^2 + \int_a^t \psi(s)h(s)ds, \ \forall t \in [a,b],$$

then

$$h(t)| \leq |c| + \int_a^t \psi(s)ds, \ \forall t \in [a,b]$$

also holds.

Denote by

$$E_1(u,t) = \varepsilon|u'(t)| + |u(t)| + \Big(\varepsilon\Big(Au(t), u(t)\Big)\Big)^{1/2} + \Big(\varepsilon\int_0^t |u'(\tau)|^2 d\tau\Big)^{1/2} +$$

$$+\Big(\int_0^t \Big(Au(\tau), u(\tau)\Big)d\tau\Big)^{1/2}.$$

LEMMA 1. *Suppose that for any $T > 0$ $f \in W^{1,1}(0,T;H), u_0, u_1 \in V$ and the operators A and B satisfy the conditions (1) and (2). Then there exists the positive constants γ and C depending on ω and L such that for the solutions of the problem (P_ε) the following estimates*

$$E_1(u,t) \leq Ce^{\gamma t}\Big(E_1(u,0) + \int_0^t \Big|f(\tau) - B(0)\Big|e^{-\gamma\tau}d\tau\Big), \quad 0 \leq t \leq T, \tag{3}$$

$$E_1(u',t) \le Ce^{\gamma t}\Big(E_1(u',0) + \int_0^t \Big|f'(\tau)\Big|e^{-\gamma\tau}d\tau\Big), \quad 0 \le t \le T \tag{4}$$

are true. If $B = 0$, then in (3) and (4) $\gamma = 0$.

Proof. Denote by

$$E(u,t) = \varepsilon^2|u'(t)|^2 + \frac{1}{2}|u(t)|^2 + \varepsilon\Big(Au(t),u(t)\Big) + \varepsilon\int_0^t |u'(\tau)|^2 d\tau +$$
$$+\varepsilon\Big(u(t),u'(t)\Big) + \int_0^t \Big(Au(\tau),u(\tau)\Big)d\tau.$$

The direct computations show that for every solution of the problem (P_ε) the following equality

$$\frac{d}{dt}E(u,t) = \Big(f(t) - B(0), u(t) + 2\varepsilon u'(t)\Big) - \Big((Bu(t)) - B(0), u(t) + \varepsilon u'(t)\Big) \tag{5}$$

is true. Since $|B(u) - B(0)| \le L|u|$, $E(u,t) \ge 0$ and $|u|(|u| + 2\varepsilon|u'|) \le 2\gamma E(u,t)$ with some $\gamma > 0$, then from (5) follows the inequality

$$\frac{d}{dt}E(u,t) \le 2\gamma E(u,t) + \Big(|f(t) - B(0)|)(|u(t)| + 2\varepsilon|u'(t)|\Big). \tag{6}$$

As $|u(t)| + 2\varepsilon|u'(t)| \le 2C(E(u,t))^{1/2}$ with some $C > 0$, then from (6) we have

$$\frac{d}{dt}\Big(e^{-2\gamma t}E(u,t)\Big) \le 2C\Big|f(t) - B(0)\Big|\Big(E(u,t)\Big)^{1/2}e^{-2\gamma t}.$$

Integrating the last inequality we obtain

$$\frac{1}{2}E(u,t)e^{-2\gamma t} \le \frac{1}{2}E(u,0) + C\int_0^t e^{-2\gamma\tau}\Big(E(u,\tau)\Big)^{1/2}\Big|f(\tau) - B(0)\Big|d\tau.$$

Using Lemma A from the last inequality we get the estimate

$$\Big(E(u,t)\Big)^{1/2} \le e^{\gamma t}\Big[\Big(E(u,0)\Big)^{1/2} + C\int_0^t \Big|f(\tau) - B(0)\Big|e^{-\gamma\tau}d\tau\Big]. \tag{7}$$

It is easy to see that there exist positive constants C_0, C_1 depending only on ω such that

$$C_0\Big(E(u,t)\Big)^{1/2} \le E_1(u,t) \le C_1\Big(E(u,t)\Big)^{1/2}. \tag{8}$$

Using the inequalities (8) from (7) we obtain the estimate (3).

To prove the estimate (4) let us denote by $u_h(t) = u(t+h) - u(t), h > 0, t \ge 0$. For any solution of the problem (P_ε) we have

$$\frac{d}{dt}E(u_h,t) = (2\varepsilon(u_h'(t) + u_h(t), f_h - (B(u(t))_h).$$

Since

$$|(B(u(t)))_h| = |B(u(t+h)) - B(u(t))| \le L|u_h(t)|, \quad |2\varepsilon u_h' + u_h(t)| \le 2C(E(u_h,t))^{1/2},$$

and

$$|2\varepsilon u_h'(t) + u_h(t)||u_h(t)| \leq 2\gamma E(u_h, t),$$

then we have

$$\frac{d}{dt}(e^{-2\gamma t}E(u_h, t)) \leq 2C(E(u_h, t))^{1/2}|f_h(t)|e^{-2\gamma t}.$$

Integrating the last inequality we get

$$E(u_h, t)e^{-2\gamma t} \leq E(u_h, 0) + \int_0^t e^{-2\gamma\tau}|f_h(\tau)|(E(u_h, \tau))^{1/2}d\tau.$$

Dividing the last inequality by h^2 and then passing to the limit as $h \to 0$ we get

$$E(u', t)e^{-2\gamma t} \leq E(u', 0) + \int_0^t e^{-2\gamma\tau}|f'(\tau)|(E(u', \tau))^{1/2}d\tau. \tag{9}$$

Since $u'(0) = u_1, \varepsilon u''(0) = f(0) - Au_0 - u_1 - B(u_0)$, then using Lemma A and (8) from (9) we obtain the estimate (4) in the same way as was obtained the the estimate (3). Lemma 1 is proved.

3. Relation between the solutions of the problems (P_ε) and (P_0) in the linear case.

In this section we shall give the relation between the solutions of the problem (P_ε) and (P_0) in the linear case, i. e. in the case when $B = 0$. This relation was inspired by the work [3]. At first we shall prove some properties of the kernel $K(t, \tau, \varepsilon)$ of transformation which realizes this connection.

For $\varepsilon > 0$ denote

$$K(t, \tau, \varepsilon) = \frac{1}{2\sqrt{\pi\varepsilon}}\Big(K_1(t, \tau, \varepsilon) + 3K_2(t, \tau, \varepsilon) - 2K_3(t, \tau, \varepsilon)\Big),$$

where

$$K_1(t, \tau, \varepsilon) = \exp\Big\{\frac{3t - 2\tau}{4\varepsilon}\Big\}\lambda\Big(\frac{2t - \tau}{2\sqrt{\varepsilon t}}\Big), \quad K_2(t, \tau, \varepsilon) = \exp\Big\{\frac{3t + 6\tau}{4\varepsilon}\Big\}\lambda\Big(\frac{2t + \tau}{2\sqrt{\varepsilon t}}\Big),$$

$$K_3(t, \tau, \varepsilon) = \exp\Big\{\frac{\tau}{\varepsilon}\Big\}\lambda\Big(\frac{t + \tau}{2\sqrt{\varepsilon t}}\Big), \quad \lambda(s) = \int_s^\infty e^{-\eta^2}d\eta.$$

LEMMA 2. [4] *The function $K(t, \tau, \varepsilon)$ possesses the following properties:*

(i) *For any fixed $\varepsilon > 0$ $K \in C(\{t \geq 0\} \times \{\tau \geq 0\}) \cap C^\infty(\{t > 0\} \times \{\tau > 0\})$;*

(ii) $K_t(t, \tau, \varepsilon) = \varepsilon K_{\tau\tau}(t, \tau, \varepsilon) - K_\tau(t, \tau, \varepsilon), \quad t > 0, \tau > 0;$

(iii) $\varepsilon K_\tau(t, 0, \varepsilon) - K(t, 0, \varepsilon) = 0, \quad t \geq 0;$

(iv) $K(0, \tau, \varepsilon) = \dfrac{1}{2\varepsilon}\exp\Big\{-\dfrac{\tau}{2\varepsilon}\Big\}, \quad \tau \geq 0;$

(v) *For each fixed $t > 0$, $s, q \in \mathbb{N}$ there exist constants $C_1(s, q, t, \varepsilon) > 0$ and $C_2(s, q, t) > 0$ such that*

$$|\partial_t^s\partial_\tau^q K(t, \tau, \varepsilon)| \leq C_1(s, q, t, \varepsilon)\exp\{-C_2(s, q, t)\tau/\varepsilon\}, \quad \tau > 0;$$

(vi) $K(t,\tau,\varepsilon) > 0, \quad t \geq 0, \quad \tau \geq 0;$

(vii) *Let ε be fixed, $0 < \varepsilon \ll 1$. For any $\varphi : [0,\infty) \to H$ continuous on $[0,\infty)$ such that $|\varphi(t)| \leq M\exp\{Ct\}$, $t \geq 0$, the relation*

$$\lim_{t\to 0}\int_0^\infty K(t,\tau,\varepsilon)\varphi(\tau)d\tau = \int_0^\infty e^{-\tau}\varphi(2\varepsilon\tau)d\tau,$$

is valid in H;

(viii) $\int_0^\infty K(t,\tau,\varepsilon)d\tau = 1, \quad t \geq 0;$

(ix) *Suppose $\rho : [0,\infty) \to \mathbb{R}$ possesses the following properties: $\rho \in C^1[0,\infty)$, ρ and ρ' are increasing functions and $|\rho(t)| \leq M\exp\{ct\}, |\rho'(t)| \leq M\exp\{ct\}$, for $t \in [0,\infty)$. Then there exist positive constants C_1 and C_2 such that*

$$\int_0^\infty K(t,\tau,\varepsilon)|\rho(t)-\rho(\tau)|d\tau \leq C_1\sqrt{\varepsilon}\exp\{C_2 t\}, \quad t > 0;$$

(x) *Let $f(t) \in W_C^{1,\infty}(0,\infty;H)$ with some $C \geq 0$. Then there exist positive constants C_1, C_2 such that*

$$\Big|f(t) - \int_0^\infty K(t,\tau,\varepsilon)f(\tau)d\tau\Big|_H \leq C_1\sqrt{\varepsilon}\exp\{C_2 t\}\|f'\|_{L_C^\infty(0,\infty;H)}, \quad t \geq 0, \quad 0 < \varepsilon \ll 1;$$

(xi) *There exists $C > 0$ such that*

$$\int_0^t\int_0^\infty K(\tau,\theta,\varepsilon)\exp\Big\{-\frac{\theta}{\varepsilon}\Big\}d\theta d\tau \leq C\varepsilon, \quad t \geq 0, \quad \varepsilon > 0.$$

Now we are ready to establish the relation between the solutions of the problem (P_ε) and the corresponding solutions of the problem (P_0) in the linear case, i. e. in the case when $B = 0$.

THEOREM 1. *Let $A : D(A) \subset H \to H$ be a linear and closed operator, $f \in L_C^\infty(0,\infty;H)$ for some $C \geq 0$. If u is a solution of the problem (P_ε) such that $u \in W_C^{2,\infty}(0,\infty;H)$ with some $C \geq 0$, then the function v_0 which is defined by*

$$v_0(t) = \int_0^\infty K(t,\tau,\varepsilon)u(\tau)d\tau$$

is a solution of the following problem:

$$\begin{cases} v_0'(t) + Av_0(t) = F_0(t,\varepsilon), \quad t > 0, \\ v_0(0) = \varphi_\varepsilon, \end{cases} \qquad (P.v_0)$$

where

$$F_0(t,\varepsilon) = \frac{1}{\sqrt{\pi}}\Big[2\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big) - \lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big]u_1 + \int_0^\infty K(t,\tau,\varepsilon)f(\tau)d\tau,$$

$$\varphi_\varepsilon = \int_0^\infty e^{-\tau} u(2\varepsilon\tau) d\tau.$$

Proof. Integrating by parts and using the properties **(i)** – **(iii)** and **(v)** of Lemma 2 we get

$$v_0'(t) = \int_0^\infty K_t(t,\tau,\varepsilon) u(\tau) d\tau = \int_0^\infty \Big(\varepsilon K_{\tau\tau}(t,\tau,\varepsilon) - K_\tau(t,\tau,\varepsilon)\Big) u(\tau) d\tau =$$

$$= \int_0^\infty K(t,\tau,\varepsilon)\Big(\varepsilon u''(\tau) + u'(\tau)\Big) d\tau + \varepsilon K(t,0,\varepsilon) u_1 - A v_0(t) + \int_0^\infty K(t,\tau,\varepsilon) f(\tau) d\tau.$$

Thus $v_0(t)$ satisfies the equation from $(P.v_0)$. From property **(viii)** of Lemma 2 follows the validity of the initial condition of $(P.v_0)$. Theorem 1 is proved.

4. Limits of the solutions of the problem (P_ε) as $\varepsilon \to 0$.

In this section we shall study the behavior of the solutions of the problem (P_ε) as $\varepsilon \to 0$.

THEOREM 2. *Suppose $f \in W_C^{1,\infty}(0,\infty;H)$, with some $C \geq 0$, $u_0, u_1 \in V$ and the operators A and B satisfy the condition (1) and (2) respectively. Then there exist positive constants C_1, C_2 such that*

$$|u(t) - v(t)| \leq C_1 M e^{C_2 t}\sqrt{\varepsilon}, \quad t \geq 0, \quad 0 \leq \varepsilon \ll 1, \tag{10}$$

where u and v are the solutions of the problems (P_ε) and $(P.v)$, respectively,

$$M = |f(0)| + |u_0| + |Au_0| + |B(u_0)| + |u_1| + ||f'||_{W_C^{1,\infty}(0,\infty;H)},$$

and C_1 and C_2 are independent on M and ε.

Proof. Under the conditions of the theorem from (4) follows the estimate

$$||u'(t)||_{L^\infty_{C_1}(0,\infty;H)} \leq CM, \quad t \geq 0. \tag{11}$$

According to Theorem 1 the function w which is defined by

$$w(t) = \int_0^\infty K(t,\tau,\varepsilon) u(\tau) d\tau$$

is a solution of the problem

$$\begin{cases} w'(t) + Aw(t) = F(t,\varepsilon), \\ w(0) = w_0, \end{cases} \tag{P.w}$$

where

$$F(t,\varepsilon) = F_0(t,\varepsilon) + \int_0^\infty K(t,\tau,\varepsilon) f(\tau) d\tau - \int_0^\infty K(t,\tau,\varepsilon) B(u(\tau)) d\tau,$$

$$F_0(t,\varepsilon) = \frac{1}{\sqrt{\pi}}\Big[2\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big) - \lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big] u_1, \quad w_0 = \int_0^\infty e^{-\tau} u(2\varepsilon\tau) d\tau.$$

Using the properties **(vi)**,**(viii)** and (**x**) of Lemma 2 and the estimate (11) we get

$$|u(t) - w(t)| \leq C e^{C_2 t}\sqrt{\varepsilon}||u'(t)||_{L^\infty_{C_1}(0,\infty;H)} \leq CM e^{C_2 t}\sqrt{\varepsilon}, \quad t \geq 0. \tag{12}$$

Let us denote $R(t) = v(t) - w(t)$, where v is the solution of the problem $(P.v)$ and w is the solution of the problem $(P.w)$. Then $R(t)$ is the solution of the problem

$$\begin{cases} R'(t) + AR(t) = B(w(t)) - B(v(t)) + \mathcal{F}(t,\varepsilon), \quad t \geq 0, \\ R(0) = R_0, \end{cases}$$

where $R_0 = u_0 - w_0$ and

$$\mathcal{F}(t,\varepsilon) = f(t) - \int_0^\infty K(t,\tau,\varepsilon) f(\tau) d\tau - F_0(t,\varepsilon) - B(w(t)) + \int_0^\infty K(t,\tau,\varepsilon) B(u(\tau)) d\tau.$$

As $R(t) \in V$ and V is continuously embedded in H then

$$(AR(t), R(t)) \geq \omega ||R(t)||^2 \geq \omega_0 |R(t)|^2, \quad \omega_0 > 0.$$

Therefore

$$\frac{d}{dt}|R(t)|^2 = -2\Big(AR(t), R(t)\Big) + 2\Big(R(t), B(w(t)) - B(v(t)\Big) + 2\Big(\mathcal{F}(t,\varepsilon), R(t)\Big) \leq$$

$$\leq 2\omega_1 |R(t)|^2 + 2|\mathcal{F}(t,\varepsilon)||R(t)|, \quad t \geq 0, \quad \omega_1 = -\omega_0 + L,$$

and hence

$$\frac{1}{2}|R(t)|^2 e^{-2\omega_1 t} \leq \frac{1}{2}|R_0|^2 + \int_0^t |\mathcal{F}(\tau,\varepsilon)||R(\tau)| e^{-2\omega_1 \tau} d\tau, \quad t \geq 0,$$

then using Lemma A we obtain the estimate

$$|R(t)| \leq e^{\omega_1 t}\Big(|R_0| + \int_0^t |\mathcal{F}(\tau,\varepsilon)| e^{-\omega_1 \tau} d\tau\Big), \quad t \geq 0. \tag{13}$$

From (11) follows the estimate

$$|R_0| \leq \int_0^\infty e^{-\tau} |u(2\varepsilon\tau) - u_0| d\tau \leq \int_0^\infty e^{-\tau} \int_0^{2\varepsilon\tau} |u'(s)| ds d\tau \leq$$

$$\leq 2\varepsilon C M \int_0^\infty \tau e^{-\tau + 2C_1 \varepsilon \tau} d\tau \leq C M \varepsilon, \quad 0 < \varepsilon \leq (4C_1)^{-1}. \tag{14}$$

Now let us estimate $|\mathcal{F}(t,\varepsilon)|$. Using the property **(x)** of Lemma 2 we have

$$\Big| f(t) - \int_0^\infty K(t,\tau) f(\tau) d\tau \Big| \leq C_1 M \sqrt{\varepsilon} e^{C_2 t}, \quad t \geq 0 \quad 0 < \varepsilon \ll 1. \tag{15}$$

As $e^\tau \lambda(\sqrt{\tau}) \leq C, \tau \geq 0$, then for $\varepsilon \in (0, (8|\omega_1|)^{-1}]$ we have

$$\int_0^t \exp\Big\{\frac{3\tau}{4\varepsilon} - \omega_1 \tau\Big\} \lambda\Big(\sqrt{\frac{\tau}{\varepsilon}}\Big) d\tau \leq \varepsilon \int_0^{\frac{t}{\varepsilon}} \exp\Big\{\frac{3\tau}{4} + |\omega_1| \tau \varepsilon\Big\} \lambda\Big(\sqrt{\tau}\Big) d\tau \leq$$

$$\leq C \int_0^\infty e^{7\tau/8} \lambda(\sqrt{\tau}) d\tau = C\varepsilon \int_0^\infty e^{-\tau/8} e^{\tau} \lambda(\sqrt{\tau}) d\tau \leq C\varepsilon, \quad t \geq 0, \quad 0 < \varepsilon \ll 1,$$

and

$$\int_0^t e^{-\omega_1 \tau} \lambda\Big(\frac{1}{2}\sqrt{\frac{\tau}{\varepsilon}}\Big) d\tau \leq \varepsilon \int_0^\infty e^{|\omega_1| \varepsilon \tau} \lambda(\frac{1}{2}\sqrt{\tau}) d\tau \leq C\varepsilon, \quad t \geq 0, \quad 0 < \varepsilon \ll 1.$$

Therefore we get the estimate

$$\int_0^t e^{-\omega_1\tau}|F_0(\tau,\varepsilon)|d\tau \le C\varepsilon|u_1| \le C\varepsilon M, \quad t\ge 0, \quad 0<\varepsilon\ll 1. \tag{16}$$

Let us estimate the difference

$$I(t)=\int_0^\infty K(t,\tau,\varepsilon)B(u(\tau))d\tau - B(w(t)) = I_1(t)+I_2(t), \tag{17}$$

where due to property (**viii**) of Lemma 2

$$I_1(t)=\int_0^\infty K(t,\tau,\varepsilon)\Big(B(u(\tau))-B(w(\tau))\Big)d\tau,$$

$$I_2(t)=\int_0^\infty K(t,\tau,\varepsilon)\Big(B(w(\tau))-B(w(t))\Big)d\tau.$$

Using condition (2), the estimate (12) and property (**ix**) of Lemma 2 we have

$$|I_1(t)| \le CLM\sqrt{\varepsilon}\int_0^\infty K(t,\tau,\varepsilon)e^{C_2\tau}d\tau \le$$

$$\le CM\sqrt{\varepsilon}\int_0^\infty K(t,\tau,\varepsilon)\Big(|e^{C_2\tau}-e^{C_2t}|+e^{C_2t}\Big)d\tau \le CM\sqrt{\varepsilon}e^{C_3t}. \tag{18}$$

To estimate $I_2(t)$ we will evaluate the function $w'(t)$. Integrating by parts and using the properties (**ii**), (**iii**) and (**iv**) of Lemma 2 we have

$$w'(t)=\int_0^\infty K_t(t,\tau,\varepsilon)u(\tau)d\tau = \int_0^\infty \Big(\varepsilon K_{\tau\tau}(t,\tau,\varepsilon)-K_\tau(t,\tau,\varepsilon)\Big)u(\tau)d\tau =$$

$$= -\int_0^\infty \Big(\varepsilon K_\tau(t,\tau,\varepsilon)-K(t,\tau,\varepsilon)\Big)u'(\tau)d\tau =$$

$$= -\frac{3}{2}\int_0^\infty K(t,\tau,\varepsilon)u'(\tau)d\tau + \frac{3}{4\sqrt{\pi\varepsilon}}\int_0^\infty \Big(K_2(t,\tau,\varepsilon)-K_3(t,\tau,\varepsilon)\Big)u'(\tau)d\tau. \tag{19}$$

Due the estimate (11) we get

$$\Big|\int_0^\infty K(t,\tau,\varepsilon)u'(\tau)d\tau\Big| \le CM\int_0^\infty K(t,\tau,\varepsilon)e^{C_2\tau}d\tau \le$$

$$\le CM\int_0^\infty K(t,\tau,\varepsilon)\Big(|e^{C_2\tau}-e^{C_2t}|+e^{C_2t}\Big)d\tau \le CMe^{C_2t}, \quad t\ge 0. \tag{20}$$

Also integrating by parts we obtain the estimates

$$\int_0^\infty K_i(t,\tau,\varepsilon)e^{C_2\tau}d\tau \le C\varepsilon e^{C_2t}, \quad i=2,3, \quad t\ge 0, \quad 0<\varepsilon\ll 1. \tag{21}$$

Using (19), (20) and (21) we get

$$|w'(t)| \le CMe^{C_3t}, \quad t\ge 0, \quad 0<\varepsilon\ll 1. \tag{22}$$

The estimate (22) and property (**ix**) of Lemma 2 permit to evaluate $I_2(t)$.

$$|I_2(t)| \leq \int_0^\infty K(t,\tau,\varepsilon)|B(w(\tau)) - B(w(t))|d\tau \leq L\int_0^\infty K(t,\tau,\varepsilon)\Big|\int_\tau^t w'(s)ds\Big|d\tau \leq$$

$$\leq LCM\int_0^\infty K(t,\tau,\varepsilon)\Big|e^{C_2 t} - e^{C_2\tau}\Big|d\tau \leq CMe^{C_3 t}\sqrt{\varepsilon}, \quad t \geq 0, \quad 0 < \varepsilon \ll 1. \qquad (23)$$

From (17), (18) and (23) we get

$$|I(t)| \leq CMe^{C_3 t}\sqrt{\varepsilon}, \quad t \geq 0, \quad 0 < \varepsilon \ll 1. \qquad (24)$$

From (15), (16) and (24) follows the estimate

$$\int_0^t e^{-\omega_1\tau}|\mathcal{F}(\tau,\varepsilon)|d\tau \leq CMe^{C_3 t}\sqrt{\varepsilon}, \quad t \geq 0, \quad 0 < \varepsilon \ll 1. \qquad (25)$$

From (13), using the estimates (14) and (25) we get

$$|R(t)| \leq C_1 Me^{C_3 t}\sqrt{\varepsilon}, \quad t \geq 0, \quad 0 < \varepsilon \ll 1. \qquad (26)$$

Finally from the estimates (12) and (26) we have

$$|u(t) - v(t)| \leq |u(t) - w(t)| + |R(t)| \leq C_1 Me^{C_3 t}\sqrt{\varepsilon}, \quad t \geq 0, \quad 0 < \varepsilon \ll 1.$$

The estimate (10) is proved.

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